Finite Groups for Family Symmetry

Jisuke Kubo
Institute for Theoretical Physics, Kanazawa University
Kanazawa 920-1192, Japan

1 Finite groups $D_N$ and $Q_N$

The dihedral symmetry is a symmetry of regular polygon. $D_N$ symmetry appears in polyatomic molecules, for instance. As we will see, the binary dihedral (dicyclic) group $Q_{2N}$ may be regarded as the covering group of $D_N$. $Q_{2N}$ has pseudo-real representations, which is welcome for chiral theories like the standard model (SM).

1.1 Definitions

The group presentation for the dihedral groups $D_N$ is given by

$$\{A_{D_N}, B_D; (A_{D_N})^N = B_D^2 = E, B_D^{-1}A_{D_N}B_D = A_{D_N}^{-1}\}$$

(1)

and

$$\{A_{Q_N}, B_Q; (A_{Q_N})^N = E, B_Q^2 = (A_{Q_N})^{N/2}, B_Q^{-1}A_{Q_N}B_Q = A_{Q_N}^{-1}\}$$

(2)

for the binary dihedral group $Q_N$, where $E$ is the identity element. For the binary dihedral group $Q_N$, $N$ should be even starting with 4, while $N$ for $D_N$ starts with 3. The $2N$ group elements are:

$$G = \{E, A, (A)^2, \ldots, (A)^{N-1}, B, AB, (A)^2B, \ldots, (A)^{N-1}B\}$$

(3)

both for $D_N$ and $Q_N$. A two-dimensional representation of $A$ and $B$ is given by

$$A_{D_N} = A_{Q_N} = \begin{pmatrix} \cos \phi_N & \sin \phi_N \\ -\sin \phi_N & \cos \phi_N \end{pmatrix} \text{ with } \phi_N = 2\pi/N,$$

(4)

$$B_D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \text{ for } D_N, \quad B_Q = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \text{ for } Q_N.$$  

(5)

Note that $\det A_{Q_N} = \det B_{Q_N} = 1$, implying that $Q_N$ is a subgroup of $SU(2)$. It follows that the dihedral group is a subgroup of $SO(3)$, which one sees if one embeds $A_{D_N}$ and $B_D$ into $3 \times 3$ matrices $[1]

$$A_{D_N} \rightarrow \begin{pmatrix} \cos \phi_N & \sin \phi_N & 0 \\ -\sin \phi_N & \cos \phi_N & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B_D \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$  

(6)

It also follows that $D_N$ has only real representations, while $Q_N$ can have real as well as pseudo-real representations. However, the smallest binary dihedral group that contains both real and pseudo-real nonsinglet representations is $Q_6$, because $Q_4$ has only pseudo-real nonsinglet representations. Note that the irreducible representations (irreps) of $D_N$ and $Q_N$ are either one- or two-dimensional.
$Q_N$ is the “double-covering group” of $D_N$ in the following sense. Consider the matrices of $D_{N/2}$, i.e., $A_{D_{N/2}}$ and $B_D$, and define $\tilde{A}_{Q_N} = A_{D_{N/2}}$, $\tilde{B}_Q = B_D$. Note that $\tilde{A}_{Q_N}$ have exactly the same properties as $A_{Q_N}$. Therefore, the set
\[
\{E, \tilde{A}_{Q_N}, (\tilde{A}_{Q_N})^2, \ldots, (\tilde{A}_{Q_N})^{N-1}, \tilde{B}_Q, \tilde{A}_{Q_N} \tilde{B}_Q, (\tilde{A}_{Q_N})^2 \tilde{B}_Q, \ldots, (\tilde{A}_{Q_N})^{N-1} \tilde{B}_Q\}
\] (7)
is a set of $Q_N$ elements. Since however $(\tilde{A}_{Q_N})^{N/2} = (A_{D_{N/2}})^{N/2} = E$ by definition, the $D_{N/2}$ elements appear twice in (7).

2 Application to the SUSY Flavor Problem [2]

Low energy supersymmetry (SUSY) is introduced to protect the Higgs mass from the quadratic divergences. Since low energy SUSY is broken, the breaking of SUSY must be soft, whatever its origin is, to maintain the very nature of low energy SUSY. Unfortunately, the most arbitrary part of a phenomenologically viable supersymmetric extension of the standard model (SM) is this soft supersymmetry breaking sector, because renormalizability allows an introduction of many soft supersymmetry breaking (SSB) parameters. In the minimal supersymmetric standard model (MSSM), more than 100 SSB parameters can be introduced. The problem is not only this large number of the SSB parameters, but also the fact that one has to highly fine tune them so that they do not induce unacceptably large flavor changing neutral currents (FCNCs) and CP violations. This problem, called the SUSY flavor problem, is not new, but has existed ever since supersymmetry found phenomenological applications.

There are several theoretical approaches to overcome this problem. In this report we consider a mechanism which is based on non-abelian discrete flavor symmetries.

2.1 $D_3(=S_3)$ model

Three generations of the quarks and leptons belong to the reducible representation of $D_3$, i.e., $3 = 1 + 2$, respectively [3, 4]. We also introduce a $D_3$ doublet Higgs pair, $H^U_I, H^D_I$ ($I = 1, 2$), as well as a $D_3$ singlet Higgs pair, $H^U_3, H^D_3$. The same R-parity is assigned to these fields as in the MSSM. Then we assume that the total superpotential is invariant under $D_3$ symmetry [2, 5]

- (i) Gaugino masses:
The gaugino masses are the same as in the MSSM.
- (ii) Trilinear couplings:
The trilinear couplings can be read off from the superpotential, from which one can obtain the soft left-right mass matrices:

\[
\tilde{m}^2_{aLR} = \begin{pmatrix}
m_a^2 A^q_1 + m_2^2 A^q_2 & m_2^2 A^q_3 & m_2^2 A^q_5 \\
m_2^2 A^q_4 & m_1^2 A^q_1 - m_2^2 A^q_2 & m_2^2 A^q_5 \\
m_1^2 A^q_4 & m_2^2 A^q_5 & m_3^2 A^q_3
\end{pmatrix}
\] (a = $\tilde{l}, \tilde{q}$), (8)

where $A^q_a$ are free parameters of dimension one.

- (ii) Soft scalar masses:
$D_3$ invariant soft scalar masses are diagonal:

\[
\tilde{m}^2_{aLL} = m_a^2 \begin{pmatrix}
a^q_L & 0 & 0 \\
0 & a^q_L & 0 \\
0 & 0 & b^q_L
\end{pmatrix}, \quad \tilde{m}^2_{aRR} = m_a^2 \begin{pmatrix}
a^q_R & 0 & 0 \\
0 & a^q_R & 0 \\
0 & 0 & b^q_R
\end{pmatrix}
\] (a = $\tilde{l}, \tilde{q}$), (9)
where \( m_{l,q} \) denote the average of the slepton and squark masses, respectively, and \((a_{L(R)}, b_{L(R)})\) are dimensionless free parameters of \( O(1) \).

We consider FCNC processes, e.g. \( Br(\mu \rightarrow e + \gamma) \), that are proportional to the off-diagonal elements of

\[
\Delta a_{LL,RR} = U_{aL,R}^\dagger \tilde{m}^2_{aLL,RR} U_{aL,R} \quad \text{and} \quad \Delta a_{LR} = U_{aL}^\dagger \tilde{m}^2_{aLR} U_{aR}. \tag{10}
\]

The experimental bounds on the dimensionless quantities

\[
\delta_{aLL,RR,LR} = \frac{\Delta a_{aLL,RR,LR}}{\tilde{m}_a^2} \quad (a = l, q), \tag{11}
\]

are known. We have computed the theoretical values of \( \delta \)'s for the present model, where

\[
\Delta a_{aL,R} = a_{aL,R}^q - b_{aL,R}^q, \quad \tilde{A}_i = \frac{A_i}{m_{\tilde{a}}} \quad (a = l, q). \tag{12}
\]

We have fond that the experimental bounds for the most of the cases are satisfied, if \( |\Delta a| \)'s and \( |(\tilde{A}_i - \tilde{A}_j)| \)'s are less than about one. The experimental constraints coming from the CP violations in the \( K^0 - \bar{K}^0 \) system on \( \delta_{12}^a \), more precisely on \( \sqrt{|\text{Im}(\delta_{12}^a)|} \), \( \sqrt{|\text{Im}(\delta_{12}^a)(\delta_{12}^a)'|} \) and \( |\text{Im}(\delta_{11}^a)| \) are very severe. Note however, one of the most strong constraint coming from \( \epsilon'/\epsilon \) on \( |\text{Im}(\delta_{12}^a)| \) is satisfied. The constraints from the electric dipole moment (EDM) of the neutron on \( |\text{Im}(\delta_{11}^a)| \) and \( |\text{Im}(\delta_{12}^a)| \) are also very severe. From the analyses in this section, we conclude that, apart from certain fine tuning [5], the FCNCs and CP phases, which are induced by the SSB parameters in \( O(1) \) disorder at \( M_{\text{SUSY}} \), are sufficiently suppressed to satisfy the experimental constraints. This is a consequence of \( D_3 (= S_3) \) flavor symmetry.

### 2.2 \( Q_6 \) model [1]

One of the successful Ansätze for the quark mass matrices is of a nearest neighbor interaction (NNI) type:

\[
M = \begin{pmatrix} 0 & C & 0 \\ \pm C & 0 & B \\ 0 & B' & A \end{pmatrix} . \tag{13}
\]

We would like to derive the mass matrix (13) solely from a symmetry principle. On finds that two conditions should be met: (i) There should be real as well as pseudo-real nonsinglet representations, and (ii) there should be the up- and down-type Higgs \( SU(2)_L \) doublets (type II Higgs). The smallest finite group that allows both real and pseudo-real nonsinglet representations is \( Q_6 \) as already found out. So, the Higgs sector of the MSSM fits the desired Higgs structure. In Table 1 we write the \( Q_6 \) assignment of the quark and lepton supermultiplets:

<table>
<thead>
<tr>
<th>( Q_6 )</th>
<th>( Q, L )</th>
<th>( U^c, D^c, E^c, N^c )</th>
<th>( H^u, H^d )</th>
<th>( Q_3, L_3 )</th>
<th>( U^c_3, D^c_3, E^c_3, N^3_3 )</th>
<th>( H^u_3, H^d_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 2 )</td>
<td>( 2' )</td>
<td>( 2' )</td>
<td>( 1' )</td>
<td>( 1'' )</td>
<td>( 1'' )</td>
<td></td>
</tr>
</tbody>
</table>

**Table 1.** \( Q_6 \) assignment of the matter supermultiplets.

We have found that CP phases can be spontaneously induced in this model. Consequently, the quark sector contains 8 real parameters with one independent phase to describe the quark masses and their mixing. Predictions in the \( |V_{ub}| - \bar{\eta}, |V_{ub}| - \sin 2\beta(\phi_1) \) and \( |V_{ub}| - |V_{td}/V_{ts}| \) planes are given in [1]. A normal as well as an inverted spectrum of neutrino masses is possible. But if one employs
another $Q_6$ assignment (as given in Table 2), one obtains exactly the same leptonic sector as the $D_3$
model with a $Z_2$ symmetry in that sector.

<table>
<thead>
<tr>
<th>$Q_6$</th>
<th>$L, E^c, N^c$</th>
<th>$L_3, E^c_3$</th>
<th>$N^c_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1$^D$</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

Table 2. an alternative $Q_6$ assignment of the quark supermultiplets is the same as in Table 1.

Because of $Q_6$ symmetry, it turns out that $R$-parity violating couplings are almost absent. Out
of the 96 $R$-parity breaking cubic couplings that are allowed in the MSSM superpotential, $Q_6$ allows
only one coupling

$$\lambda'[(L_1Q_2 + L_2Q_1)D^c_1 + (L_1Q_1 - L_2Q_2)D^c_2].$$

(14)

Many couplings vanish because of color antisymmetry and $SU(2)_L$ antisymmetry. Furthermore, all
baryon number violating cubic terms are forbidden by $Q_6$ alone. This means that there is no proton
decay problem in the present model.

As for the SUSY flavor problem, we may expect that $Q_6$ suppresses strongly FCNC and CP
violating processes that are induced by the SSB terms. However, the constraints coming from the
EDM of neutron, electron and mercury atom are very severe, as we have seen in the $D_3$ model.
For instance, $(\delta_{11})_{LR}$ has to satisfy $|\text{Im}(\delta^d_{11})_{LR}| < 6.7 \times 10^{-8}(\tilde{m}_q/100\text{ GeV})^2$. Similar constraints
exist for $(\delta^u_{11})_{LR}$ and $(\delta^e_{11})_{LR}$, too. (The quantity $\delta^d_{LR}$ is defined in (11).) Since the CP phases
can be spontaneously induced in the $Q_6$ model, and thanks to $Q_6$ symmetry the mass matrix $\tilde{m}^2_{LR}$
has exactly the same structure as the mass matrix of the matter supermultiplets, we conclude that
$(\delta^d_{11})_{LR}$ is a real number. From the same reason, all $\delta_{LR}$ are real; phase alignment occurs. Thus, we
can satisfy the most stringent constraint on the $A$ terms without any fine tuning. This is true not
only at a particular energy scale, but also for the entire energy scale, which should be compared with
the case of the MSSM.

References

(2003).