Finite Groups for Family Symmetry

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1 Finite groups D_N and Q_N

The dihedral symmetry is a symmetry of regular polygon. D_N symmetry appears in polyatomic molecules, for instance. As we will see, the binary dihedral (dicyclic) group Q_{2N} may be regarded as the covering group of D_N. Q_{2N} has pseudo-real representations, which is welcome for chiral theories like the standard model (SM).

1.1 Definitions

The group presentation for the dihedral groups D_N is given by

$$\{A_{D_N}, B_D; (A_{D_N})^N = B_D^2 = E,\ B_D^{-1} A_{D_N} B_D = A_{D_N}^{-1}\}, \tag{1}$$

and

$$\{A_{Q_N}, B_Q; (A_{Q_N})^N = E, B_Q^2 = (A_{Q_N})^{N/2}, B_Q^{-1} A_{Q_N} B_Q = A_{Q_N}^{-1}\} \tag{2}$$

for the binary dihedral group Q_N, where E is the identity element. For the binary dihedral group Q_N, N should be even starting with 4, while N for D_N starts with 3. The $2N$ group elements are:

$$\mathcal{G} = \{E, A, (A)^2, \ldots, (A)^{N-1}, B, AB, (A)^2B, \ldots, (A)^{N-1}B\} \tag{3}$$

both for D_N and Q_N. A two-dimensional representation of A and B is given by

$$A_{D_N} = A_{Q_N} = \begin{pmatrix} \cos\phi_N & \sin\phi_N \\ -\sin\phi_N & \cos\phi_N \end{pmatrix} \text{ with } \phi_N = 2\pi/N, \tag{4}$$

$$B_D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ for } D_N,\ B_Q = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \text{ for } Q_N. \tag{5}$$

Note that $\det A_{Q_N} = \det B_{Q_N} = 1$, implying that Q_N is a subgroup of $SU(2)$. It follows that the dihedral group is a subgroup of $SO(3)$, which one sees if one embeds A_{D_N} and B_D into 3×3 matrices [1]

$$A_{D_N} \to \begin{pmatrix} \cos\phi_N & \sin\phi_N & 0 \\ -\sin\phi_N & \cos\phi_N & 0 \\ 0 & 0 & 1 \end{pmatrix}, B_D \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{6}$$

It also follows that D_N has only real representations, while Q_N can have real as well as pseudo-real representations. However, the smallest binary dihedral group that contains both real and pseudo-real nonsinglet representations is Q_6, because Q_4 has only pseudo-real nonsinglet representations. Note that the irreducible representations (irreps) of D_N and Q_N are either one- or two-dimensional.

Q_N is the "double-covering group" of D_N in the following sense. Consider the matrices of $D_{N/2}$, i.e., $A_{D_{N/2}}$ and B_D, and define $\tilde{A}_{Q_N} = A_{D_{N/2}}$, $\tilde{B}_Q = B_D$. Note that $\tilde{A}_{Q_N}$ have exactly the same properties as A_{Q_N}. Therefore, the set

$$\{E, \tilde{A}_{Q_N}, (\tilde{A}_{Q_N})^2, \ldots, (\tilde{A}_{Q_N})^{N-1}, \tilde{B}_Q, \tilde{A}_{Q_N}\tilde{B}_Q, (\tilde{A}_{Q_N})^2\tilde{B}_Q, \ldots, (\tilde{A}_{Q_N})^{N-1}\tilde{B}_Q\} \tag{7}$$

is a set of Q_N elements. Since however $(\tilde{A}_{Q_N})^{N/2} = (A_{D_{N/2}})^{N/2} = E$ by definition, the $D_{N/2}$ elements appear twice in (7).

2 Application to the SUSY Flavor Problem [2]

Low energy supersymmetry (SUSY) is introduced to protect the Higgs mass from the quadratic divergences. Since low energy SUSY is broken, the breaking of SUSY must be soft, whatever its origin is, to maintain the very nature of low energy SUSY. Unfortunately, the most arbitrary part of a phenomenologically viable supersymmetric extension of the standard model (SM) is this soft supersymmetry breaking sector, because renormalizability allows an introduction of many soft supersymmetry breaking (SSB) parameters. In the minimal supersymmetric standard model (MSSM), more than 100 SSB parameters can be introduced. The problem is not only this large number of the SSB parameters, but also the fact that one has to highly fine tune them so that they do not induce unacceptably large flavor changing neutral currents (FCNCs) and CP violations. This problem, called the SUSY flavor problem, is not new, but has existed ever since supersymmetry found phenomenological applications.

There are several theoretical approaches to overcome this problem. In this report we consider a mechanism which is based on non-abelian discrete flavor symmetries.

2.1 $D_3(= S_3)$ model

Three generations of the quarks and leptons belong to the reducible representation of D_3, i.e., $\mathbf{3} = \mathbf{1}+\mathbf{2}$, respectively [3, 4]. We also introduce a D_3 doublet Higgs pair, $H_I^U, H_I^D (I = 1, 2)$, as well as a D_3 singlet Higgs pair, H_3^U, H_3^D. The same R-parity is assigned to these fields as in the MSSM. Then we assume that the total superpotential is invariant under D_3 symmetry [2, 5]

The SSB sector consists of:
(i) Gaugino masses:
The gaugino masses are the same as in the MSSM.
(ii) Trilinear couplings:
The trilinear couplings can be read off from the superpotential, from which one can obtain the soft left-right mass matrices:

$$\tilde{\mathbf{m}}^{\mathbf{2}}_{aLR} = \begin{pmatrix} m_1^a A_1^a + m_2^a A_2^a & m_2^a A_2^a & m_5^a A_5^a \\ m_2^a A_2^a & m_1^a A_1^a - m_2^a A_2^a & m_5^a A_5^a \\ m_4^a A_4^a & m_4^a A_4^a & m_3^a A_3^a \end{pmatrix} \quad (a = \tilde{l}, \tilde{q}), \tag{8}$$

where A_i^a are free parameters of dimension one.
(ii) Soft scalar masses:
D_3 invariant soft scalar masses are *diagonal*:

$$\tilde{\mathbf{m}}^{\mathbf{2}}_{aLL} = m_a^2 \begin{pmatrix} a_L^a & 0 & 0 \\ 0 & a_L^a & 0 \\ 0 & 0 & b_L^a \end{pmatrix}, \ \tilde{\mathbf{m}}^{\mathbf{2}}_{aRR} = m_a^2 \begin{pmatrix} a_R^a & 0 & 0 \\ 0 & a_R^a & 0 \\ 0 & 0 & b_R^a \end{pmatrix} \quad (a = \tilde{l}, \tilde{q}), \tag{9}$$

where $m_{\tilde{l},\tilde{q}}$ denote the average of the slepton and squark masses, respectively, and $(a_{L(R)}, b_{L(R)})$ are dimensionless free parameters of $O(1)$.

We consider FCNC processes, *e.g.* $Br(\mu \to e + \gamma)$, that are proportional to the off-diagonal elements of

$$\Delta^a_{LL,RR} = U^\dagger_{aL,R}\, \tilde{\mathbf{m}}^{\mathbf{2}}_{aLL,RR}\, U_{aL,R} \text{ and } \Delta^a_{LR} = U^\dagger_{aL}\, \tilde{\mathbf{m}}^{\mathbf{2}}_{aLR}\, U_{aR}. \tag{10}$$

The experimental bounds on the dimensionless quantities

$$\delta^a_{LL,RR,LR} = \Delta^a_{LL,RR,LR}/m^2_{\tilde{a}} \quad (a = l, q), \tag{11}$$

are known. We have computed the theoretical values of δ's for the present model, where

$$\Delta a^a_{L,R} = a^a_{L,R} - b^a_{L,R},\ \tilde{A}^a_i = \frac{A^a_i}{m_{\tilde{a}}} \quad (a = l, q). \tag{12}$$

We have fond that the experimental bounds for the most of the cases are satisfied, if $|\Delta a|$'s and $|(\tilde{A}_i - \tilde{A}_j)|$'s are less than about one. The experimental constraints coming from the CP violations in the $K^0 - \bar{K}^0$ system on δ^d_{12}, more precisely on $\sqrt{|\mathrm{Im}(\delta^d_{12})^2_{LL,RR}|}$, $\sqrt{|\mathrm{Im}(\delta^d_{12})_{LL}(\delta^d_{12})_{RR}|}$ and $|\mathrm{Im}(\delta^d_{12})_{LR}|$ are very severe. Note however, one of the most strong constraint coming from ϵ'/ϵ on $|\mathrm{Im}(\delta^d_{12})_{LR}|$ is satisfied. The constraints from the electric dipole moment (EDM) of the neutron on $|\mathrm{Im}(\delta^d_{11})_{LR}|$ and $|\mathrm{Im}(\delta^u_{11})_{LR}|$ are also very severe. From the analyses in this section, we conclude that, apart from certain fine tuning [5], the FCNCs and CP phases, which are induced by the SSB parameters in $O(1)$ disorder at M_{SUSY}, are sufficiently suppressed to satisfy the experimental constraints. This is a consequence of $D_3(= S_3)$ flavor symmetry.

2.2 Q_6 model [1]

One of the successful Ansätze for the quark mass matrices is of a nearest neighbor interaction (NNI) type:

$$M = \begin{pmatrix} 0 & C & 0 \\ \pm C & 0 & B \\ 0 & B' & A \end{pmatrix}. \tag{13}$$

We would like to derive the mass matrix (13) solely from a symmetry principle. On finds that two conditions should be met: (i) There should be real as well as pseudo-real nonsinglet representations, and (ii) there should be the up- and down-type Higgs $SU(2)_L$ doublets (type II Higgs). The smallest finite group that allows both real and pseudo-real nonsinglet representations is Q_6 as already found out. So, the Higgs sector of the MSSM fits the desired Higgs structure. In Table 1 we write the Q_6 assignment of the quark and lepton supermultiplets:

	Q, L	U^c, D^c, E^c, N^c	H^u, H^d	Q_3, L_3	U^c_3, D^c_3, E^c_3, N_3	H^u_3, H^d_3
Q_6	$\mathbf{2}$	$\mathbf{2'}$	$\mathbf{2'}$	$\mathbf{1'}$	$\mathbf{1'''}$	$\mathbf{1'''}$

Table 1. Q_6 assignment of the matter supermultiplets.

We have found that CP phases can be spontaneously induced in this model. Consequently, the quark sector contains 8 real parameters with one independent phase to describe the quark masses and their mixing. Predictions in the $|V_{ub}| - \bar{\eta}$, $|V_{ub}| - \sin 2\beta(\phi_1)$ and $|V_{ub}| - |V_{td}/V_{ts}|$ planes are given in [1]. A normal as well as an inverted spectrum of neutrino masses is possible. But if one employes another Q_6 assignment (as given in Table 2), one obtains exactly the same leptonic sector as the D_3 model with a Z_2 symmetry in that sector.

	L, E^c, N^c	L_3, E_3^c	N_3^c
Q_6	$\mathbf{2}$	$\mathbf{1''}$	$\mathbf{1}$

Table 2. *analternative*Q_6 assignment of the quark supermultiplets is the same as in Table 1.

Because of Q_6 symmetry, it turns out that R-parity violating couplings are almost absent. Out of the 96 R-parity breaking cubic couplings that are allowed in the MSSM superpotential, Q_6 allows only one coupling

$$\lambda'[(L_1Q_2 + L_2Q_1)D_1^c + (L_1Q_1 - L_2Q_2)D_2^c]. \tag{14}$$

Many couplings vanish because of color antisymmetry and$SU(2)_L$ antisymmetry. Furthermore, all baryon number violating cubic terms are forbidden by Q_6 alone. This means that there is no proton decay problem in the present model.

As for the SUSY flavor problem, we may expect that Q_6 suppresses strongly FCNC and CP violating processes that are induced by the SSB terms. However, the constraints coming from the EDM of neutron, electron and mercury atom are very severe, as we have seen in the D_3 model. For instance, $(\delta_{11}^d)_{LR}$ has to satisfy $|\mathrm{Im}(\delta_{11}^d)_{LR}| < 6.7 \times 10^{-8}(\tilde{m}_q/100\ \mathrm{GeV})^2$. Similar constraints exist for $(\delta_{11}^u)_{LR}$ and $(\delta_{11}^e)_{LR}$, too. (The quantity δ_{LR}^d is defined in (11).) Since the CP phases can be spontaneously induced in the Q_6 model, and thanks to Q_6 symmetry the mass matrix $\tilde{\mathbf{m}}^2_{dLR}$ has exactly the same structure as the mass matrix of the matter supermultiplets, we conclude that $(\delta_{11}^d)_{LR}$ is a real number. From the same reason, all δ_{LR} are real; phase alignment occurs. Thus, we can satisfy the most stringent constraint on the A terms without any fine tuning. This is true not only at a particular energy scale, but also for the entire energy scale, which should be compared with the case of the MSSM.

References

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